Journal of Engineering Physics and Thermophysics, Vol. 65, No. 6, 1993

# REGULARIZATION OF NONSTATIONARY PROBLEMS FOR ELLIPTIC EQUATIONS

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UDC 519.63

*We consider properties and prove stationarity of regularized approximations of one of the problems with reverse time for a hyperbolic equation. Via time reversal, the latter is converted to an elliptic equation.*

An extensive class of science and engineering problems reduces to solving incorrect problems for elliptic equations. In the form of the problem of extending the solutions of elliptic equations and the Cauchy problem, we formulate some inverse problems of thermophysics, mechanical engineering, prospecting geophysics, plasma physics, etc. For example, in a number of cases the determination of the temperature in a certain region and on a part of its boundary with a redefinition of the conditions on the other part (the temperature and the heat flux are specified) can be reduced to solving the Cauchy problem for an elliptic equation (the stationary boundary-value inverse problem [1]). Problems of this type are incorrect in the classical sense because of the violation of the condition of solution stationarity relative to the disturbance of additional conditions generally prescribed with an error (for example, errors of the temperature measurement at the region boundary). Nonstationarity generates great difficulties at the stage of problem solution, and therefore it is urgent to devise stationary methods of solution, substantiate their regularizing properties, and compare various approaches. As applied to the Cauchy problem for an elliptic equation, the present work considers the regularizing properties of two approaches: the use of a nonlocal boundary condition and the variational formulation of the initial problem. We obtained, in particular, the equivalence conditions for the solutions of nonlocal and extremal problems.

**Statement of the Problem.** Let $D$ denote a bounded region of the $n$-dimensional space $R^n$ and $\partial D$, the boundary of the region. In $R^n \times \{ -\infty < t < \infty \}$ we consider the finite cylinder $Q_T = \{(x, t) \mid x \in D, 0 < t < T\}$ with the lateral surface $\Gamma = \{(x, t) \mid x \in \partial D, 0 < t < T\}$, where $x = (x_1, x_2, ..., x_n\}$. We define for $x \in D$ the elliptic operator

$$Au = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a_0 (x) u . \tag{1}$$

Everywhere below, for the operator $A$ and the convex region $D$ we assume the following conditions to be fulfilled:

$$\mu_1 \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq \mu_2 \sum_{i=1}^{n} \xi_i^2 , \quad 0 < a_0 < \mu_3 ,$$
$$a_{ij} = a_{ji} ; \quad a_0 , \quad a_{ij} \in C^1 (D) , \quad \partial D \in C^2 , \quad \mu_i > 0 , \quad i = 1 , 2 . \tag{2}$$

In the region $Q_T$ we consider the problem of obtaining $u(x, t)$ from the equation

$$\frac{\partial^2 u}{\partial t^2} - Au = 0 , \quad (x , t) \in Q_T \tag{3}$$

with the homogeneous boundary conditions

$$u (x , t) = 0 , \quad (x , t) \in \Gamma \tag{4}$$

Moscow State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 65, No. 6, pp. 690-694, December, 1993.

and Cauchy conditions of the form

$$\frac{\partial u}{\partial t}(x, 0) = 0, \quad x \in D, \tag{5}$$

$$u(x, 0) = \varphi(x), \quad x \in D. \tag{6}$$

Among the approaches to solving the nonstationary problem of evolutionary type (3)-(6) it is possible to separate: construction of regularized difference schemes [3-5], a quasiinversion method [5, 6], a variational formulation of the initial problem [7], and use of a disturbance of the Cauchy conditions that is nonlocal with respect to $t$ [8-10].

We examine the following extremal formulation of problem (3)-(5): to find a minimum of the smoothing functional

$$J(z) = \| u(x, 0) - \varphi^{\delta}(x) \|^2 + \alpha \| z(x) \|^2 \tag{7}$$

with the constraint

$$u(x, T) = z(x), \quad x \in D \tag{8}$$

and with conditions (3)-(5) fulfilled for $u(x, t)$. In the expression for $J(z)$, $\alpha > 0$ is a regularization parameter [2] and $\| \cdot \|$ is a norm in $L_2(D)$. In addition, Eq. (7) takes into account that, instead of condition (6) with a precisely specified function $\varphi$, we have

$$u(x, 0) = \varphi^{\delta}(x), \quad x \in D, \tag{9}$$

where $\varphi^{\delta}$ is a disturbed function, for which the following expression holds true:

$$\| \varphi(x) - \varphi^{\delta}(x) \| \leq \delta \tag{10}$$

($\delta$ is the error level). Employing functionals (7) is one of the ways of attaining solution stationarity for many incorrect problems [2]. The first term in the right side of Eq. (7) is a so-called discrepancy functional [1, 2], and the second is a stabilizing (smoothing) term.

It turned out that the problem with a nonlocal boundary condition utilizing a deflection of the values of the function $u(x, t)$ from $t = 0$ to a certain time $T^* \geq T$ is closely adjacent to an extremal problem. Here we consider problem (3)-(5) with the nonlocal condition

$$u(x, 0) + \beta u(x, T^*) = c\varphi^{\delta}(x), \quad T^* \geq T, \quad x \in D, \tag{11}$$

where $\beta$ is a regularization parameter and $c$ is a constant close in magnitude to unity.

**Equivalence of the Solutions of Nonlocal and Extremal Problems.** To obtain the conditions of equivalence of the solutions of the two formulated problems we use their representation as a Fourier expansion. Let eigenvalues of the operator $A$ be denoted by $\lambda_k$, $k = 1, 2, \ldots$, $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$, and $\omega_k$ be the corresponding orthonormal system of eigenfunctions. Let $(\cdot, \cdot)$ be the scalar product in $L_2(D)$ with the norm

$$\| f \|^2 = (f, f) = \int_D f^2 dx.$$

It is easy to verify that the operator $R(T, \alpha)$ that gives the solution $z(x) = R(T, \alpha)\varphi^{\delta}(x)$ of problem (3)-(6) has the form

$$R(T, \alpha) = \frac{\operatorname{ch}(T\sqrt{A})}{1 + \alpha \operatorname{ch}^2(T\sqrt{A})} \tag{12}$$

or

$$z(x) = \sum_{k=1}^{\infty} \frac{\operatorname{ch}(T\sqrt{\lambda_k})}{1 + \alpha \operatorname{ch}^2(T\sqrt{\lambda_k})} \varphi_k^\delta \omega_k(x),$$

where $\varphi_k^\delta = (\varphi^\delta, \omega_k)$. We point out immediately that, in the obtained representation of the solution $z(x)$ of the problem of minimizing the smoothing functional, higher harmonics ($\lambda_k \to \infty$) are "cut off" due to the stabilizing factor $\gamma_k = (1 + \alpha \operatorname{ch}^2(T\sqrt{\lambda_k}))^{-1}$. This behavior of $z(x)$ permits us to count on solution stationarity.

The corresponding operator that gives the solution of the problem with the nonlocal condition (3)-(5) and (11) has the form

$$R(T, \alpha) = c \frac{\operatorname{ch}(T\sqrt{A})}{1 + \beta \operatorname{ch}(T^* \sqrt{A})}. \tag{12'}$$

Comparing expressions (12) and (12') by direct checking, it is possible to be convinced that the following assertion is valid.

T H E O R E M 1 (of equivalence). The solution of the extremal problem (3)-(5), (7), and (8) coincides with the solution of the nonlocal problem (3)-(5) and (11) at $T^* = 2T$, $c = 1/(1 + \alpha/2)$, and $\beta = \alpha/(2 + \alpha)$.

Thus, it is established that the extremal problem and the problem with a nonlocal condition are reducible to one another. This fact may be employed both in the theoretical study of the indicated problems and in their numerical solution.

**Regularizing Properties of Extremal and Nonlocal Problems.** We denote by $w_{2,0}^2(D)$ the subspace of $w_2^2(D)$ in which all functions twice continuously differentiable in $\bar{D}$ and equal to zero on $\partial D$ are a dense set. Under constraints (2) imposed on the operator $A$ and the region $D$, the expression $\| u \|_2^2 = (Au, Au)$ is equivalent to the norm in $w_{2,0}^2(D)$ and any function $u$ from this space is expanded into the series

$$u(x) = \sum_{k=1}^{\infty} u_k \omega_k(x), \quad u_k = (u, \omega_k),$$

that converges to $u(x)$ in the norm of $w_{2,0}^2(D)$ [11]. Here the equality

$$\| u(x) \|_2^2 = \sum_{k=1}^{\infty} \lambda_k^2 u_k^2.$$

is true.

T H E O R E M 2. Let $\varphi, \varphi^\delta \in L_2(D)$, $\| \varphi - \varphi^\delta \| \le \delta$, and the exact solution $u(x, T) = R(T, 0)\varphi(x)$ of problem (3)-(6) be bounded in $w_{2,0}^2(D)$: $\| u(x, T) \|_2 \le M$. Then, as $\delta \to 0$ and $\alpha(\delta) \to 0$, the solution $z(x)$ of problem (3)-(5), (7), and (8) converges to $u(x, T)$ in $w_{2,0}^2(D)$ and the stationarity evaluation

$$\| z \|_2 \le \frac{8}{\alpha e^2 T^2} \| \varphi^\delta \|. \tag{13}$$

is valid.

P r o o f. We first prove evaluation (13). Taking into consideration Eq. (11), we examine the expression

$$\| z \|_2^2 = \| R(T, \alpha)\varphi^\delta \|_2^2) = \sum_{k=1}^{\infty} \lambda_k^2 \frac{\operatorname{ch}^2(T\sqrt{\lambda_k})}{(1 + \alpha \operatorname{ch}^2(T\sqrt{\lambda_k}))^2} (\varphi^\delta \omega_k)^2 \le$$

$$\le \sum_{k=1}^{\infty} \frac{\lambda_2^2}{\alpha^2 \operatorname{ch}^2(T\sqrt{\lambda_k})} (\varphi^\delta \omega_k)^2 \le \sum_{k=1}^{\infty} \frac{4}{\alpha^2} \frac{\lambda_k^2}{\exp(2T\sqrt{\lambda_k})} (\varphi^\delta \omega_k)^2 \le$$

$$\leq \frac{4}{\alpha^2}\frac{\lambda_0^2}{\exp(2T\sqrt{\lambda_0})}\sum_{k=1}^{\infty}(\varphi^{\delta}\omega_k)^2 = \frac{4}{\alpha^2}\frac{\lambda_0^2}{\exp(2T\sqrt{\lambda_0})}\|\varphi^{\delta}\|^2,$$

where $\lambda_0 = 4/T^2$. Thus, operator (12) generated by the solution of the extremal problem (3)-(5), (7), and (8) is bounded in the norm of the space $w_{2,0}^2(D)$.

We examine further the difference of approximate and exact solutions in the norm $\|\cdot\|_2$:

$$\begin{aligned}\|z(x) - u(x, T)\|_2 &= \|R(T, \alpha)(\varphi^{\delta} - \varphi) + (R(T, \alpha) - R(T, 0))\varphi\|_2 \leq \\ &\leq \|R(T, \alpha)(\varphi - \varphi^{\delta})\|_2 + \|(R(T, \alpha) - R(T, 0))\varphi\|_2 \leq \\ &\leq \frac{8}{e^2T^2\alpha}\|\varphi - \varphi^{\delta}\| + \|(R(T, \alpha) - R(T, 0))\varphi\|_2 \leq \\ &\leq \frac{8\delta}{e^2T^2\alpha} + \|(R(T, \alpha) - R(T, 0))\varphi\|_2.\end{aligned} \tag{14}$$

We then assess the closeness of the operator $R(T, \alpha)$ to the operator of the exact solution $R(T, 0)$ of problem (3)-(6):

$$d = \|(R(T, \alpha) - R(T, 0))\varphi\|_2^2 =$$

$$= \sum_{k=1}^{\infty}\lambda_k^2\,\mathrm{ch}^2(T\sqrt{\lambda_k})\left(1 - \frac{1}{1 + \alpha\,\mathrm{ch}^2(T\sqrt{\lambda_k})}\right)^2\varphi_k^2.$$

By virtue of the boundedness of the exact solution $u(x, T)$ of problem (3)-(6) in the norm of $w_{2,0}^2(D)$ it can be asserted that, for any $\varepsilon > 0$, there is an $r(\varepsilon)$ such that

$$\sum_{k=r(\varepsilon)}^{\infty}\lambda_k^2\,\mathrm{ch}^2(T\sqrt{\lambda_k})\,\varphi_k^2 \leq \varepsilon^2/8.$$

Then

$$d \leq \sum_{k=1}^{r(\varepsilon)}\lambda_k^2\,\mathrm{ch}^2(T\sqrt{\lambda_k})\,\varphi_k^2\left(1 - \frac{1}{1 + \alpha\,\mathrm{ch}^2(T\sqrt{\lambda_k})}\right)^2 +$$

$$+ \sum_{k=r(\varepsilon)+1}^{\infty}\lambda_k^2\,\mathrm{ch}^2(T\sqrt{\lambda_k})\,\varphi_k^2 \leq M^2\sum_{k=1}^{r(\varepsilon)}\left(1 - \frac{1}{1 + \alpha\,\mathrm{ch}^2(T\sqrt{\lambda_k})}\right)^2 + \varepsilon^2/8.$$

For each $r(\varepsilon)$ it is possible to indicate an $\alpha_0$ such that, for $\alpha < \alpha_0$,

$$M^2\sum_{k=1}^{r(\varepsilon)}\left(1 - \frac{1}{1 + \alpha\,\mathrm{ch}^2(T\sqrt{\lambda_k})}\right)^2 < \varepsilon^2/8.$$

Hence,

$$\|(R(T, \alpha) - R(T, 0))\varphi\|_2 \leq \left(\frac{\varepsilon^2}{8} + \frac{\varepsilon^2}{8}\right)^{1/2}. \tag{15}$$

From Eqs. (14) and (15) we derive

$$\| z(x) - u(x, T) \|_2 \leq \frac{8\delta}{e^2 T^2 \alpha} + \frac{\varepsilon}{2}.$$

It is evident that there is an $\alpha(\delta) < \alpha_0$ such that at any $\varepsilon > 0$ there is a sufficiently small $\delta(\varepsilon)$ for which

$$\frac{8\delta}{e^2 T^2 \alpha} \leq \frac{\varepsilon}{2}$$

and then

$$\| z(x) - u(x, T) \|_2 \leq \varepsilon.$$

Thus the theorem is proved.

So, it is demonstrated that with $\delta \to 0$ and provided the regularization parameter $\alpha$ matches the error $\delta$ ($\alpha = \alpha(\delta)$), the solution $z(x) = R(T, \alpha)\varphi^\delta(x)$ converges to the exact solution $u(x, T)$ of problem (3)-(6) in the norm of the space $w_{2,0}^2(D)$. In other words, the operator $R(T, \alpha)$ of (12) generated by problem (3)-(5), (7), and (8) is regularizing for the Cauchy problem in the class of functions $w_{2,0}^2(D)$.

Note 1. Under the hypotheses of the theorem, instead of boundedness of $u(x,T) \in w_{2,0}^2(D)$ ($\| u(x, T) \|_2 \leq M$) it is possible to require boundedness in $L_2 D \times (\| u(x, T) \| \leq M)$. Under the assumption of greater smoothness of $\varphi(x)$, i.e., $\varphi(x) \in w_{2,0}^2(D)$ instead of $\varphi(x) \in L_2(D)$, the assertion of the theorem remains the same.

Note 2. If the dimension of the space is $n \leq 3$, then the operator $R(T, \alpha)$ of (12) will be regularizing in the class of continuous functions due to the continuity of the imbedding of the space $w_2^2(D)$ in $C(\overline{D})$ [11].

By virtue of equivalence of the solutions that is established in Theorem 1, the results of Theorem 2 extend also to the problem with a nonlocal condition (3)-(5) and (11).

A direct solution of the problems formulated above brings about the problem of selecting the regularization parameter $\alpha(\beta)$. For the problem of minimizing the functional (7), by analogy with [2] we may substantiate the choice of $\alpha$ from the condition

$$\| u(x, 0) - \varphi^\delta(x) \| = \delta$$

(the choice of $\alpha$ by the discrepancy),where $u(x, 0)$ is the solution of the direct problem (3)-(5) and (8) with $z(x)$ obtained as a result of solving the extremal problem. Because the solutions of the extremal and nonlocal problems are equivalent, this way of selecting $\alpha$ is applicable also to the problem with a nonlocal condition (3)-(5) and (11).

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